$\mathrm{U}(1)_A$ Symmetry and Correlation Functions in the High Temperature Phase of QCD

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Simple group-theoretical arguments are used to demonstrate that in the high temperature (chirally restored) phase of QCD with N_f massless flavours, all n-point correlation functions of quark bilinears are invariant under $\mathrm{U}(1)_A$ transformations provided $n < N_f$. In particular this implies that the two-point correlation function in the η' channel is identical to that in the pion channel for $N_f > 2$. Unlike previous work, this result does not depend on the topological properties of QCD and can be formulated without explicit reference to functional integrals.

The nature of the high temperature phase of QCD is a problem of considerable importance. The approximate $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ chiral symmetry of QCD in this phase is believed to be broken only by non-zero quark masses. The study of this phase is both intrinsically interesting and of relevance to cosmology and ultra-relativistic heavy ion collisions. One aspect of the problem that has received recent interest [1–5] is the role of the anomalously broken $\mathrm{U}(1)_A$ symmetry in the chirally restored phase.

From a theoretical perspective, the natural way to study the possible role of $\mathrm{U}(1)_A$ symmetry breaking in the high temperature phase of QCD is to study correlation functions of composite operators constructed from quark and gluon fields and to compare multi-point correlation functions that are related by $\mathrm{U}(1)_A$ symmetry (and perhaps also $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ symmetry). All observable manifestations of anomalous $\mathrm{U}(1)_A$ symmetry breaking should be reflected in the behaviour of these correlation functions. It has been suggested by Shuryak that the $\mathrm{U}(1)_A$ symmetry might be restored along with $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ in the high temperature phase, in the sense that no $\mathrm{U}(1)_A$-violating effects can be found among correlation functions in this phase [1]. Moreover, it has been shown that unless there are contributions from configurations in the functional integral that form a set of measure zero in the chiral limit, $\mathrm{U}(1)_A$-violating correlation functions must vanish [3].

On the other hand, Lee and Hatsuda (LH) [4] and Evans, Hsu and Schwetz (EHS) [5] have argued that contributions from the winding-number-one sector do not vanish and are exactly the kind of zero-measure contributions that exploit the loophole in ref. [3]. From studying the form of the functional determinant, LH and EHS conclude that for QCD with two light flavours the effects of $\mathrm{U}(1)_A$ violation can be seen in the study of two-point functions of quark bilinears. However, for three or more light flavours they conclude that $\mathrm{U}(1)_A$ violation cannot be observed in two-point functions. For example, disregarding explicit symmetry breaking due to the quark masses, the correlation functions in the π, σ and η' channels are identical. This in turn means that the screening masses m_π, m_σ and $m_{\eta'}$ are all degenerate. This may seem startling at first glance, since it has been well known since 't Hooft's seminal papers [6] that the anomaly together with topology solves the $\mathrm{U}(1)_A$ problem, allowing the η' to be split from the pion. If LH and EHS are correct, however, for $N_f \geq 3$ the splitting of the π from the η' for $m_q = 0$ depends on spontaneous breaking of the chiral symmetry. They argue more generally that n-point correlations of quark bilinears will have no $\mathrm{U}(1)_A$-violating effects for $n < N_f$ but can, and in general will, have observable $\mathrm{U}(1)_A$ violation for $n \geq N_f$.

The arguments of LH and EHS are highly suggestive but not definitive. In particular, the argument depends on taking the infinite-volume limit after the chiral limit, which might not be permissible since issues associated with symmetry breaking often depend on taking the infinite-volume limit first. However, as we will show in this letter, the conclusion of EHS that all $\mathrm{U}(1)_A$-violating n-point correlation functions of quark bilinears vanish for $n < N_f$ is, in fact, correct. This includes the somewhat counter-intuitive result that the π and η' channels are degenerate above T_c for three or more flavours. The proof given here is quite simple and relies only on basic group theory; it does not depend explicitly on topological properties of QCD.

Before addressing the problem in its general form, we will turn our attention to a very simple demonstration that in the chiral limit the π and η' two-point correlation functions are degenerate above T_c for three flavour QCD. For simplicity of notation, we denote pseudoscalar quark-bilinear composite fields as ϕ_a and scalar quark bilinears as ξ_a, with subscripts $a = 0$ for the singlets and $a = 1, \ldots N_f^2 - 1$ for the rest (such as the pion triplet in SU(2) or the octet in SU(3)). Thus for example the quark bilinear $\overline{q}(x) i\gamma_5 \lambda_a q(x)$ is denoted $\phi_a(x)$, where in addition to the $N_f^2 - 1$ generalised Gell-Mann matrices we have introduced λ_0, defined as $\sqrt{2/N_f}$ times the unit matrix.

Suppose, taking $N_f = 3$, we consider a two-point correlation function in a pionic channel—for example the ϕ_3 correlator:

$$\langle \overline{q}(x) i\gamma_5 \lambda_3 q(x) \; \overline{q}(0) i\gamma_5 \lambda_3 q(0)\rangle \equiv \langle \phi_3(x)\, \phi_3(0)\rangle \quad . \tag{1}$$

Now consider what happens to this correlator under a particular SU(3) axial transformation

$$q \;\to\; q' \;=\; e^{i\gamma_5\left(\sqrt{3}\lambda_8 - \lambda_3\right)\frac{\pi}{4}}\, q \quad . \tag{2}$$

It is simple to show that under this transformation ϕ_3 transforms in the following way

$$\phi_3 \;\to\; \phi_3' \;=\; \sqrt{\frac{2}{3}}\phi_0 + \sqrt{\frac{1}{3}}\phi_8 \quad . \tag{3}$$

Thus, under the axial rotation in eq. (2) the correlation function in eq. (1) transforms according to

$$\begin{aligned}\langle \phi_3(x)\phi_3(0)\rangle \;\to\; \langle \phi_3'(x)\phi_3'(0)\rangle \;=\; &\frac{2}{3}\langle \phi_0(x)\phi_0(0)\rangle \;+\; \frac{1}{3}\langle \phi_8(x)\phi_8(0)\rangle \\ &+ \frac{\sqrt{2}}{3}\langle \phi_0(x)\phi_8(0)\rangle \;+\; \frac{\sqrt{2}}{3}\langle \phi_8(x)\phi_0(0)\rangle \quad .\end{aligned} \tag{4}$$

Neglecting explicit $\mathrm{SU}(3)_V$ breaking due to quark masses, all thermal expectation values must be $\mathrm{SU}(3)_V$ singlets since flavour symmetry is not spontaneously broken. This means that the last two terms of eq. (4) are identically zero. Moreover $\mathrm{SU}(3)_V$ symmetry implies that the ϕ_3 correlator must be equal to ϕ_8 correlator. Thus, eq. (4) can be rewritten as

$$\langle \phi_3'(x)\phi_3'(0)\rangle \;=\; \frac{2}{3}\langle \phi_0(x)\phi_0(0)\rangle \;+\; \frac{1}{3}\langle \phi_3(x)\phi_3(0)\rangle \quad . \tag{5}$$

If one is studying the chirally restored phase, as we are here, then by definition all correlation functions are invariant under arbitrary $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ transformations, including eq. (2). Thus $\langle \phi_3(x)\phi_3(0)\rangle = \langle \phi_3'(x)\phi_3'(0)\rangle$, which along with eq. (5) implies that $\langle \phi_3(x)\phi_3(0)\rangle = \langle \phi_0(x)\phi_0(0)\rangle$. In other words the two-point correlation function in the pion channel is identical to the correlation function in the η' channel. This completes the demonstration. For more than three flavours, it is easy to generalise the preceding argument, with the same result. However, it should be noted that the argument does not work for two flavours, since there is no analogue of the λ_8 rotation which is essential to obtain eq. (3).

Now let us consider the problem more generally. We will use the fact that only $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$-symmetric multipoint correlation functions have a thermal expectation value in the restored phase to restrict severely the number of possible independent n-point functions for low n, and show that for $n < N_f$ they are all invariant under $\mathrm{U}(1)_A$ transformations.

The quark bilinears transform under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ as the direct sum of $(N_f, \overline{N_f})$ and $(\overline{N_f}, N_f)$, as can be seen by writing, for instance, $\phi_3 \equiv \overline{q} i\gamma_5\lambda_3 q$ as $i(q_L^\dagger \gamma_0 \lambda_3 q_R - q_R^\dagger \gamma_0 \lambda_3 q_L)$. From them we can form linear combinations that have definite transformation properties, namely

$$\mathbf{M} = \sum_{a=0}^{N_f^2-1} (\xi_a + i\phi_a)\lambda_a \quad : \quad (N_f, \overline{N_f}) \tag{6}$$

$$\mathbf{M}^\dagger = \sum_{a=0}^{N_f^2-1} (\xi_a - i\phi_a)\lambda_a \quad : \quad (\overline{N_f}, N_f) \tag{7}$$

Under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ transformations $\mathbf{M}$ and $\mathbf{M}^\dagger$ transform as $\mathbf{M} \to \mathbf{U}_L \mathbf{M} \mathbf{U}_R^\dagger$ and $\mathbf{M}^\dagger \to \mathbf{U}_R \mathbf{M}^\dagger \mathbf{U}_L^\dagger$. We will also need the parity transformation, $\mathbf{M} \to \mathbf{M}^\dagger$, and the behaviour under $\mathrm{U}(1)_A$ rotations through θ, $\mathbf{M} \to e^{i\theta}\mathbf{M}$.

There are two differences between $N_f = 2$ and $N_f > 2$: firstly, $\overline{2} \equiv 2$ so that $\mathbf{M}$ and $\mathbf{M}^\dagger$ have the same tensor structure, and secondly the symmetric structure constants d_{abc} vanish for SU(2). As a result there are two independent chiral multiplets $(\xi_0, \boldsymbol{\phi})$ and $(\phi_0, \boldsymbol{\xi})$ (better known as $(\sigma, \boldsymbol{\pi})$ and $(\eta', \boldsymbol{\delta})$). However for $N_f > 2$, no such separation occurs and all $2N_f^2$ mesons transform into one another under a general chiral transformation.

In the restored phase of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, the only vacuum correlators which can be non-vanishing are $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ singlets ("chiral singlets") that are also even under parity. (We are neglecting effects due to finite current-quark masses which explicitly break the symmetry.) Thus none of the bilinears have vacuum expectation values—for

instance, $\langle \overline{q}(x)q(x)\rangle = 0$. Chiral singlets can be constructed in two distinct ways. One is to take equal numbers of $\mathbf{M}$'s and $\mathbf{M}^\dagger$'s, coupled up to a singlet. Examples are $\mathrm{Tr}\,(\mathbf{M}_1^\dagger \mathbf{M}_2)$, $\mathrm{Tr}\,(\mathbf{M}_1^\dagger \mathbf{M}_2 \mathbf{M}_3^\dagger \mathbf{M}_4)$, $\mathrm{Tr}\,(\mathbf{M}_1^\dagger \mathbf{M}_2)\mathrm{Tr}\,(\mathbf{M}_3^\dagger \mathbf{M}_4)$, etc. (where $\mathbf{M}_i \equiv \mathbf{M}(x_i)$). Only even-parity combinations have vacuum expectation values, giving one independent two-point function $\mathrm{Tr}\,(\mathbf{M}_1^\dagger \mathbf{M}_2) + \mathrm{Tr}\,(\mathbf{M}_2^\dagger \mathbf{M}_1)$, twelve independent four-point functions (six of which involve products of two-point functions) and so on. All of these are not only chiral invariants; they are obviously $\mathrm{U}(1)_A$-invariant as well.

The other way of obtaining a singlet is to couple N_f $\mathbf{M}$'s or N_f $\mathbf{M}^\dagger$'s together. This produces two singlets,

$$\frac{1}{(N_f)!}\epsilon_{ijk\ldots p}\epsilon_{i'j'k'\ldots p'}(\mathbf{M}_1)_{ii'}(\mathbf{M}_2)_{jj'}(\mathbf{M}_3)_{kk'}\ldots(\mathbf{M}_{N_f})_{pp'} \tag{8}$$

and the analogous expression with $\mathbf{M}\to\mathbf{M}^\dagger$. (All indices run from 1 to N_f.) For identical $\mathbf{M}$'s, these terms are just $\det\mathbf{M}$ and $\det\mathbf{M}^\dagger$. By parity, only the sum has a non-vanishing vacuum expectation value. This, however, is *not* $\mathrm{U}(1)_A$-invariant. Further chiral singlet, $\mathrm{U}(1)_A$-violating terms may be obtained by coupling, for instance, $N_f + 1$ $\mathbf{M}$'s and an $\mathbf{M}^\dagger$, etc.

The crucial point is that $\mathrm{U}(1)_A$-violating terms can only be obtained from structures such as eq. (8) involving at least N_f bilinears, so for $n < N_f$ all n-point functions in the chirally restored phase are $\mathrm{U}(1)_A$-invariant, completely independently of the strength of the anomaly.

For $N_f = 2$, there are two chiral-singlet two-point functions with even parity, one $\mathrm{U}(1)_A$-invariant and one $\mathrm{U}(1)_A$-violating. If the non-perturbative effects of the anomaly persist above the chiral restoration point, the $\eta' - \eta'$ and $\pi - \pi$ correlators will be different. However for $N_f = 3$ or higher, there is only one, $\mathrm{U}(1)_A$-invariant, chiral singlet, and the $\eta' - \eta'$ and $\pi - \pi$ correlators must be equal. In other words, the η' and pion will be degenerate, as was already shown above.

The above arguments are easily extended to more general quark bilinears. Vector and axial-vector bilinears, $\overline{q}\gamma_\mu\lambda_a q$ and $\overline{q}i\gamma_\mu\gamma_5\lambda_a q$, are themselves $\mathrm{U}(1)_A$ singlets, and so their correlators can never violate $\mathrm{U}(1)_A$. The tensor bilinears can be grouped into three sets, $(\overline{q}\sigma^{0i}\lambda_a q,\ \frac{1}{2}\epsilon_{ijk}\overline{q}\sigma^{jk}\lambda_a q)$, for $i = 1{-}3$. For each i, these transform under $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R$ and $\mathrm{U}(1)_A$ exactly like the scalars and pseudoscalars (ξ_a, ϕ_a). Thus the arguments above can be repeated exactly, with the same conclusion: the lowest $\mathrm{U}(1)_A$-violating correlators are N_f-point functions.

The notation we have used is that of the N_f-flavour linear sigma model [7], and of course similar arguments have long been used in constructing the N_f invariants of that model, which are $\mathrm{Tr}[(\mathbf{M}^\dagger\mathbf{M})^n]$, $n < N_f$, and $(\det\mathbf{M}+\det\mathbf{M}^\dagger)$. However it must be stressed that all that the two have in common is the group structure, and the arguments presented here are independent of any dynamical model.

These results have implications for lattice studies of screening masses in the chirally restored phase of QCD. Calculations showing degeneracy of the pion and delta screening masses would give unequivocal proof of $\mathrm{U}(1)_A$ restoration only if the symmetry group is $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. It is not clear that any current lattice technique reproduces this.

In summary, we have shown on purely group-theoretic grounds that, in the chirally restored phase of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, $\mathrm{U}(1)_A$ violation cannot occur in correlation functions of n quark bilinears if $n < N_f$. Thus for $N_f > 2$, lattice studies of screening masses in mesonic channels cannot determine whether $\mathrm{U}(1)_A$ is also restored. This conclusion is the same as that of refs. [4,5], but our argument is general and does not make reference to the topology of the QCD vacuum.

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